Week 10

Last week:

Substitution in integration $\leftrightarrow$ Chain rule in differentiation

eg $\frac{d}{dx}\left[\sin(\ln|x|)\right]^3$

differentiate layer by layer

$= 3\left[\sin(\ln|x|)\right]^2\cos(\ln|x|)\frac{1}{x}$

eg $\int 3\left[\sin(\ln|x|)\right]^2\cos(\ln|x|)\frac{1}{x}\,dx$

$= \int 3\left[\sin(\ln|x|)\right]^2\cos(\ln|x|)\,d(\ln|x|)$

$= \int 3\left[\sin(\ln|x|)\right]^2\,d\sin(\ln|x|)$

$= \sin(\ln|x|)^3 + C$

integrate layer by layer

Rationalization

eg $\int \frac{\sqrt{x}}{x+1}\,dx$

let $u = \sqrt{x} = x^{\frac{1}{2}}$

$du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2}u^{-1}dx$

$\Rightarrow dx = 2u\,du$

$\int \frac{\sqrt{x}}{x+1}\,dx = \int \frac{u}{u^2+1}(2u\,du)$

$= 2\int \frac{u^2}{u^2+1}\,du$ $\qquad u^2 = u^2+1-1$

$= 2\int\left(1 - \frac{1}{u^2+1}\right)du$

$= 2(u - \arctan u) + C$

$= 2\sqrt{x} - 2\arctan\sqrt{x} + C$

eg $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$

Sol $\sqrt{x} = x^{\frac{1}{2}} = u^3$

$\sqrt[3]{x} = x^{\frac{1}{3}} = u^2$

Let $u = x^{\frac{1}{6}}$

$u^6 = x$

$6u^5\,du = dx$

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx = \int \frac{u^3}{1+u^2}\,6u^5\,du = 6\int \frac{u^8}{1+u^2}\,du$$

Long division:

$\Rightarrow u^8 = (u^2+1)(u^6-u^4+u^2-1)+1$

$\Rightarrow \frac{u^8}{u^2+1} = u^6-u^4+u^2-1+\frac{1}{u^2+1}$

$$\Rightarrow \int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx = 6\int\left(u^6-u^4+u^2-1+\frac{1}{u^2+1}\right)du$$

$$= 6\left[\frac{u^7}{7}-\frac{u^5}{5}+\frac{u^3}{3}-u+\arctan u\right]+C$$

$$= \frac{6}{7}x^{\frac{7}{6}}-\frac{6}{5}x^{\frac{5}{6}}+2x^{\frac{1}{2}}-6x^{\frac{1}{6}} + \arctan x^{\frac{1}{6}}+C$$

Integration by parts

$$d(uv) = (du)v + u(dv)$$

$$u\,dv = d(uv) - v\,du$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

eg $\int xe^x dx$ $\qquad \dfrac{de^x}{dx} = e^x$

$$= \int x\,de^x \qquad e^x dx = de^x$$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

eg $\int x \sin x\,dx$

Sol

(Bad integration by parts)

$\int x \sin x\,dx$ $\qquad \dfrac{dx^2}{dx} = 2x$

$$= \frac{1}{2}\int \sin x\,dx^2$$

$$= \frac{1}{2}\left(x^2 \sin x - \int x^2 d\sin x\right)$$

$$= \frac{1}{2}\left(x^2 \sin x - \int x^2 \cos x\,dx\right)$$

higher power (even worse)

In this computation

integrate $x \rightsquigarrow \frac{1}{2}x^2$

differentiate $\sin x \rightsquigarrow \cos x$

(Good integration by parts)

$$\int x \sin x\,dx$$

$$= -\int x\,d\cos x$$

lower power good!

$$= -\left(x\cos x - \int \cos x\,dx\right)$$

$$= -(x\cos x - \sin x) + C$$

$$= \sin x - x\cos x + C$$

In this computation

integrate $\sin x \rightsquigarrow -\cos x$

differentiate $x \rightsquigarrow 1$

eg $\int \arccos x \, dx$

$= (\arccos x)(x) - \int x \, d(\arccos x)$

$= (\arccos x)(x) - x\left(-\frac{1}{\sqrt{1-x^2}}\right)dx$

$= (\arccos x)(x) + \int \frac{x}{\sqrt{1-x^2}} dx$

$= (\arccos x)(x) + \frac{1}{2}\int \frac{dx^2}{\sqrt{1-x^2}}$

$= (\arccos x)(x) - \frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}}$

$= (\arccos x)(x) - \frac{1}{2} \cdot 2\sqrt{1-x^2} + C$

$= (\arccos x)(x) - \sqrt{1-x^2} + C$

Some Guideline for integration by parts of

$$\boxed{\int x^n f(x)\, dx}$$

$\boxed{\text{Case 1}}$: If $f(x) = \sin x, \cos x$ or e^x

integrate $f(x)$, differentiate x^n

$\boxed{\text{Case 2}}$: If $f(x) = \arcsin x, \arccos x, \ln x$

integrate x^n, differentiate $f(x)$

eg $\int \underset{\text{int}}{x^n} \underset{\text{diff}}{\ln x}\, dx = \frac{1}{n+1}\int \ln x \, dx^{n+1}$

Case 2

$= \frac{1}{n+1}\left[x^{n+1}\ln x - \int x^{n+1} d\ln x\right]$

$= \frac{1}{n+1}\left[x^{n+1}\ln x - \int x^n dx\right]$

$= \frac{1}{n+1}\left[x^{n+1}\ln x - \frac{1}{n+1}x^{n+1}\right] + C$

$= \frac{x^{n+1}}{(n+1)^2}\left[(n+1)\ln x - 1\right] + C$

eg $\int e^{ax}\cos bx\,dx$ $a,b\neq 0$ are constants

example where "LHS appears on RHS"

Sol let $I=\int e^{ax}\cos bx\,dx$

$$I=\frac{1}{b}\int e^{ax}\,d\sin bx$$

$$=\frac{1}{b}\left[e^{ax}\sin bx-\int \sin bx\,de^{ax}\right]$$

not easier than I

$$=\frac{1}{b}\left[e^{ax}\sin bx-a\underline{\int e^{ax}\sin bx\,dx}\right]$$

Try integration by parts once more

$$=\frac{1}{b}e^{ax}\sin bx+\frac{a}{b^2}\int e^{ax}\,d\cos bx$$

I!

$$=\frac{1}{b}e^{ax}\sin bx+\frac{a}{b^2}\left[e^{ax}\cos bx-a\underline{\int e^{ax}\cos bx\,dx}\right]$$

$$=\frac{1}{b}e^{ax}\sin bx+\frac{a}{b^2}e^{ax}\cos bx-\frac{a^2}{b^2}I+C$$

$$\Rightarrow I=\left(\frac{1}{b}e^{ax}\sin bx+\frac{a}{b^2}e^{ax}\cos bx\right)\Big/\left(1+\frac{a^2}{b^2}\right)+C'$$

$\leftarrow$ Just another constant $\left(C'=\frac{C}{1+\frac{a^2}{b^2}}\right)$

eg $\int\cos(\ln x)\,dx$

Sol let $u=\ln x$ $du=\frac{1}{x}dx$

$\Rightarrow dx=x\,du=e^u\,du$

$$\therefore \int\cos(\ln x)\,dx$$

$$=\int\cos u\,e^u\,du$$

$$=\int e^u\cos u\,du$$

$$=\frac{e^u\sin u+e^u\cos u}{2}+C$$

(by last example)

$$=\frac{x\sin(\ln x)+x\cos(\ln x)}{2}+C$$

Ex $\int x^2 \sin x \, dx$

$$\boxed{\text{Ans: } -x^2\cos x + 2x\sin x + 2\cos x + C}$$

Hint: Apply integration by parts twice

Reduction formula

$$\int x^n e^{ax}\,dx, \quad n \geq 0 \quad a \neq 0$$

Goal: lower degree of x.

using integration by parts repeatly

Sol Let $I_n = \int x^n e^{ax}\,dx$

then
$$I_n = \frac{1}{a}\int x^n e^{ax}\,d(ax)$$
$$= \frac{1}{a}\int x^n \, de^{ax}$$
$$= \frac{1}{a}\left(x^n e^{ax} - \int e^{ax}\,dx^n\right)$$
$$= \frac{1}{a}\left(x^n e^{ax} - n\int x^{n-1} e^{ax}\,dx\right)$$
$$= \frac{1}{a}\left(x^n e^{ax} - n I_{n-1}\right)$$

eg $I_2 = ?$

$$I_0 = \int e^{ax}\,dx = \frac{1}{a}e^{ax} + C_0$$
$$I_1 = \frac{1}{a}\left(xe^{ax} - I_0\right) = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax} + C_1$$
$$I_2 = \frac{1}{a}\left(x^2 e^{ax} - 2I_1\right) = \frac{1}{a}x^2 e^{ax} - \frac{2}{a^2}xe^{ax} + \frac{2}{a^3}e^{ax} + C_2$$

eg $I_n = \int \frac{1}{x^n(1+x)}\,dx \quad n \geq 0$

Trick: $1 = (1+x) - x$

$$I_n = \int \left(\frac{1+x}{x^n(1+x)} - \frac{x}{x^n(1+x)} \right) dx$$

$$= \int \left(\frac{1}{x^n} - \frac{1}{x^{n-1}(1+x)} \right) dx$$

$$\begin{cases} \frac{1}{1-n}\,\frac{1}{x^{n-1}} - I_{n-1} & \text{if } n \geq 2 \\ \ln|x| - I_0 & \text{if } n = 1 \end{cases}$$

$$I_0 = \int \frac{1}{1+x}\,dx = \ln|1+x| + C$$

$$I_1 = \ln|x| - I_0 = \ln|x| - \ln|1+x| + C$$

$$I_2 = -\frac{1}{x} - I_1 = -\frac{1}{x} - \ln|x| + \ln|1+x| + C$$

eg $\int \tan^m x \sec^n x\,dx, \quad m, n \geq 0$

Last week: m is odd and/or n is even (substitution)

This time: m is even and n is odd (integration by parts)

Case I $m = 0$ Let $I_n = \int \sec^n x\,dx$

$$I_n = \int \sec^{n-2}x \cdot \sec^2 x\,dx = \int \sec^{n-2} x\,d\tan x$$

$$= (\sec^{n-2}x)\tan x - \int \tan x\,d(\sec^{n-2}x)$$

$$= \sec^{n-2}x \tan x - (n-2)\int \tan x \sec^{n-3}x\,(\tan x \sec x)\,dx$$

$$= \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2}x \tan^2 x\,dx$$

$$= \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2}x(\sec^2 x - 1)\,dx$$

$$= \sec^{n-2}x \tan x - (n-2)\int (\sec^n x - \sec^{n-2}x)\,dx$$

$$= \sec^{n-2}x \tan x - (n-2)(I_n - I_{n-2})$$

$$(n-1)I_n = \sec^{n-2}x\tan x + (n-2)I_{n-2}$$

$$\Rightarrow \boxed{I_n = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}I_{n-2}}$$

$$I_1 = \int \sec x\,dx$$

$$= \ln|\sec x + \tan x| + C$$

$$I_3 = \frac{1}{2}\sec x\tan x + \frac{1}{2}I_1$$

$$= \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

$I_5, I_7, I_9 \cdots$ can be computed

Case II m is even, n is odd $m=2k$

$$\int \tan^{2k}x\sec^n x\,dx$$

$$= \int (\tan^2 x)^k \sec^n x\,dx$$

$$= \int \underbrace{(\sec^2 x - 1)^k \sec^n x}\,dx$$

Terms of odd power of $\sec x$

$\rightsquigarrow$ Reduce to Case I

Partial fractions

Goal: Express a proper rational function as sum of "simpler" proper function

Rmk $\frac{p(x)}{q(x)}$ is proper if $\deg p(x) < \deg q(x)$

eg $\frac{x}{x^2+3x+2} = \frac{-1}{x+1} + \frac{2}{x+2}$

$$\frac{x^2+20x+11}{(x+1)^2(x-3)} = \frac{-4}{x+1} + \frac{2}{(x+1)^2} + \frac{5}{x-3}$$

$$\frac{4x^2+14x-9}{(x^2+x+1)(x-2)} = \underbrace{\frac{-x+7}{x^2+x+1} + \frac{5}{x-2}}$$

Rmk: RHS is easier for integration

Recall:

A quadratic polynomial ax^2+bx+c is irreducible

$\Leftrightarrow \Delta = b^2-4ac < 0$ i.e, cannot be factorized into product of real linear factors

- $x^2-1=(x+1)(x-1)$ is reducible $\Delta=4$
- x^2+x+10 is irreducible $\Delta=-39$

Procedure Given proper $\frac{p(x)}{q(x)}$

① Factorize $q(x)$ into product of linear and irreducible quadratic factors

② Write down general terms

Factor of $q(x)$	Terms in partial fractions
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
ax^2+bx+c $\Delta=b^2-4ac<0$	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

③ Determine the unknown coefficients A_i, B_i in step ② by $\underbrace{\text{substitution}}_{\text{usually easier}}$ or comparing coefficents.

eg 1 $\dfrac{9x-13}{x^2+x-12}$

Sol

① $x^2+x-12=(x+4)(x-3)$

② General term:

Let $\dfrac{9x-13}{x^2+x-12}=\dfrac{A}{x+4}+\dfrac{B}{x-3}$ $\forall x$

③ $\dfrac{9x-13}{(x+4)(x-3)}=\dfrac{A(x-3)+B(x+4)}{(x+4)(x-3)}$

$$9x-13 = Ax-3A+Bx+4B$$
$$= (A+B)x+(-3A+4B)$$

Compare coefficients $\Rightarrow \begin{cases} A+B=9 & \cdots (i) \\ -3A+4B=-13 & \cdots (ii) \end{cases}$

$3\times(i)+(ii) \Rightarrow 7B=14 \Rightarrow B=2$

Put $B=2$ into (i) $\Rightarrow A=7$

$\therefore \dfrac{9x-13}{x^2+x-12}=\dfrac{7}{x+4}+\dfrac{2}{x-3}$

eg 2 $\dfrac{x^2+20x+11}{(x+1)^2(x-3)}$ ← already factorized

Sol

Let $\dfrac{x^2+20x+11}{(x+1)^2(x-3)}=\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}+\dfrac{C}{x-3}$

$\Rightarrow x^2+20x+11 = A(x+1)(x-3)+B(x-3)+C(x+1)^2$

Put $x=3 \Rightarrow 80=16C \Rightarrow C=5$

Put $x=-1 \Rightarrow -8=-4B \Rightarrow B=2$

Put $x=0 \Rightarrow 11=-3A-3B+C=-3A-1$

$\Rightarrow A=-4$

$\therefore \dfrac{x^2+20x+11}{(x+1)^2(x-3)}=\dfrac{-4}{x+1}+\dfrac{2}{(x+1)^2}+\dfrac{5}{x-3}$

eg 3 $\dfrac{4x^2+14x-9}{(x^2+x+1)(x-2)}$

Sol $\underbrace{x^2+x+1}$

$\Delta = b^2-4ac = -3 < 0$

$\Rightarrow x^2+x+1$ is irreducible

let $\dfrac{4x^2+14x-9}{(x^2+x+1)(x-2)} = \dfrac{Ax+B}{x^2+x+1} + \dfrac{C}{x-2}$

$4x^2+14x-9 = (Ax+B)(x-2) + C(x^2+x+1)$

Substitution eg: $x = 2, 0, 1$

or

Equating coefficients

$\Rightarrow \begin{cases} A = -1 \\ B = 7 \\ C = 5 \end{cases}$

$\Rightarrow \dfrac{4x^2+14x-9}{(x^2+x+1)(x-2)} = \dfrac{-x+7}{x^2+x+1} + \dfrac{5}{x-2}$

eg 4

$\dfrac{x^4+3x^2-x+1}{x(x^2+1)^2}$

Sol let $\dfrac{x^4+3x^2-x+1}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2}$

Solving for A, B, C, D, E

get $\dfrac{x^4+3x^2-x+1}{x(x^2+1)^2} = \dfrac{1}{x} + \dfrac{x-1}{(x^2+1)^2}$

Rmk If $\dfrac{p(x)}{q(x)}$ is improper ($\deg p \geq \deg q$), then we have to apply long division first:

eg $\dfrac{2x^3}{x^2-1}$

$$\begin{array}{r} 2x \\ x^2-1 \overline{)\, 2x^3+0x^2+0x+0} \\ \underline{2x^3 -2x } \\ 2x \end{array}$$

$2x^3 = 2x(x^2-1) + 2x$

$\dfrac{2x^3}{x^2-1} = 2x + \dfrac{2x}{x^2-1} = 2x + \dfrac{1}{x-1} + \dfrac{1}{x+1}$